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A New Subgraph of Minimum Weight Triangulations*

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Abstract. In this paper, two sufficient conditions for identifying a subgraph of minimum weight triangulation of a planar point set are presented. These conditions are based on local geometric properties of an edge to be identified. Unlike the previous known sufficient conditions for identifying subgraphs, such as Keil's β-skeleton and Yang and Xu's double circles, The local geometric requirement in our conditions is not necessary symmetric with respect to the edge to be identified. The identified subgraph is different from all the known subgraphs including the newly discovered subgraph: so-called the intersection of local-optimal triangulations by Dickerson et al. An $O(n^3)$ time algorithm for finding this subgraph from a set of n points is presented.

Keywords:

1. Introduction

Let $S = \{s_i \mid i = 0, \ldots, n-1\}$ be a set of n points in a plane. For simplicity, we assume that S is in general position so that no three points in S are colinear. Let $\overline{s_i s_j}$ for $i \neq j$ denote the line segment with endpoints s_i and s_j, and let $\omega(s_i s_j)$ denote the weight of $\overline{s_i s_j}$, that is the Euclidean distance between s_i and s_j.

A *triangulation* of S, denoted by $T(S)$, is a maximum set of non-crossing line segments with their endpoints in S. It follows that the interior of the convex hull of S is partitioned into non-overlapping triangles. The weight of a triangulation $T(S)$ is given by

$$\omega(T(S)) = \sum_{\overline{s_i s_j} \in T(S)} \omega(s_i s_j).$$

A *minimum weight triangulation*, denoted by *MWT*, of S is defined as for all possible $T(S)$, $\omega(MWT(S)) = \min\{\omega(T(S))\}$.

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$MWT(S)$ is one of the outstanding open problems listed in Garey and Johnson's book (1979). The complexity status of this problem is unknown since 1975 [SH75]. A great deal of works has been done to seek the ultimate solution of the problem. Basically, there are two directions to attack the problem. The first one is to identify edges inclusive or exclusive to $MWT(S)$ (Cheng and Xu, 1995; Keil, 1994; Yang et al., 1994) and the second one is to construct exact $MWT(S)$ for restricted classes of point set (Anagnostou and Corneil, 1993; Cheng et al., 1995; Gilbert, 1979; Klinesek, 1980). In the first direction, two subdirections have been taken. It is obvious that the intersection of all possible $T(S)$s is a subgraph of $MWT(S)$. Recently, Dickerson and Montague (1996) have shown that the intersection of all local optimal triangulations of S is a subgraph of $MWT(S)$. (A triangulation $T(S)$ is called k-gon local optimal, denoted by $T_k(S)$, if any k-sided simple polygon extracted from $T(S)$ is an optimal triangulation for this k-gon by those edges of $T(S)$ lying inside this k-gon.) Then, if the $MWT(S)$ is unique, then the following inclusion property holds:

$$\bigcap T(S) \subseteq \bigcap T_4(S) \subseteq \cdots \subseteq \bigcap T_{n-1}(S) \subseteq MWT(S)$$

This approach has some flavor of global consideration when k is increased, however, it seems difficult to find the intersections as k is increased.

Gilbert (1979) showed that the shortest edge in S is in $MWT(S)$. Yang et al. (1994) showed that mutual nearest neighbors are also in $MWT(S)$. Keil (1994) presented that the so-called β-skeleton of S for $\beta = \sqrt{2}$ is a subgraph of $MWT(S)$. Cheng and Xu (1996) extended Keil's result to $\beta = 1.17682$. The edge identification of $MWT(S)$ seems to be a promising approach and has the following merits. The more edges of $MWT(S)$ being identified, the less disconnected components is S. Thus, it is possible that eventually all these identified edges form a connected graph so that an $MWT(S)$ can be constructed by dynamic programming in polynomial time (Cheng et al., 1995). Moreover, it has been shown in Xu and Zhou (1995) that the increase of the size of subgraph of $MWT(S)$ could improve the performance of some heuristics.

The second direction is to construct exact $MWT(S)$ for restricted classes of point set S. Gilbert (1979) and Klinesek (1980) independently showed an $O(n^3)$ time dynamic programming algorithm to obtain an $MWT(S)$, where S is restricted to a simple n-gon. Recently, Anagnostou and Corneil (1993) gave an $O(n^{3k+1})$ time algorithm to find an $MWT(S)$, where S is restricted on k nested convex polygons. Meijer and Rappaport (1993) later improved the bound to $O(n^k)$ when S is restricted on k non-intersecting lines. At the same time, it was shown in Cheng et al. (1995) that if given a subgraph of $MWT(S)$ with k connected components, then the complete $MWT(S)$ can be computed in $O(n^{k+2})$ time.

This paper can be classified as the first direction. The paper is organized as follows. Section 2 surveys the recent results in this direction. Section 3 presents our sufficient conditions. Section 4 proposes an algorithm for finding a subgraph of $MWT(S)$ using the given sufficient conditions. Finally, we conclude our work.

2. A review of previous approaches

A trivial subgraph of the $MWT(S)$ is the convex hull of S, $CH(S)$, since it exists in any $MWT(S)$. A simple extension of the above idea is the intersection of all possible triangulations of S called *stable line segments* (Mirzain et al., 1996), denoted by SL(S), such that

$$SL(S) = \bigcap_{T(S)\in J} T(S),$$

where J denotes the set of all possible triangulations of S. The structure properties and the algorithms for finding $SL(S)$ were discussed in (Mirzain et al., 1996).

A recent result obtained by Dickerson and Montague (1996) showed that a subgraph $LOT(S)$ of $\cap T_4(S)$ can be found in $O(n^4)$ time and $O(n^3)$ space. The following inclusion relation holds:

$$CH(S) \subseteq SL(S) \subseteq LOT(S) \subseteq \bigcap T_4(S).$$

Another class of subgraphs of $MWT(S)$ was identified using some local geometric properties related to an edge (Cheng and Xu, 1996; Keil, 1994; Yang et al., 1994). Keil first pointed out an inclusion condition for an edge in $MWT(S)$, so-called β-skeleton.

Fact 1 (Keil, 1994). *If x and y are the endpoints of an edge in the $\sqrt{2}$-skeleton of S, and p, q, r, and s are four distinct points in S other than x and y with p and s lying on one side and q and r on the other of the line extending $\overline{xy}$. Assume $\overline{pq}$ and $\overline{rs}$ cross $\overline{xy}$, and $\overline{pq}$ does not intersect $\overline{rs}$. Then, either $|\overline{pq}| > |\overline{qr}|$ or $|\overline{rs}| > |\overline{qr}|$. (Refer to part (a) of figure* 1.)

With the above Fact (called *remote length lemma*), Keil proved that if the shaded disks are empty of points of S, $\overline{xy}$ is an edge of any $MWT(S)$. Thus, $\sqrt{2}$-skeleton(S) is a subgraph

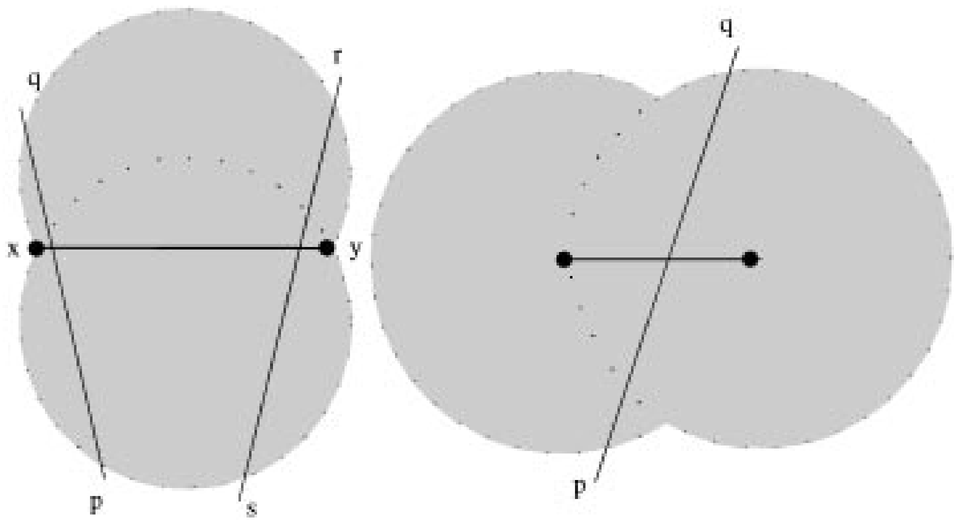


Figure 1. An illustration for the remote lemma of Keil (a) and YXY's double circles (b).

of $MWT(S)$ which can be found in $O(n \log n)$ time and $O(n)$ space. The β-value was strengthened to 1.17682 in (Cheng and Xu, 1996).

Yang et al. (1994) showed that the mutual nearest neighbours in S are subgraph of an $MWT(S)$ and their result can be stated as follows.

Fact 2 (Yang et al., 1994). *If any edge $\overline{pq}$ intersecting $\overline{xy}$ for $p, q, x, y \in S$ satisfies the following inequality*:

$$\omega(xy) \leq \min\{\omega(px), \omega(py), \omega(qx), \omega(qy)\},$$

then $\overline{xy}$ is in any $MWT(S)$.

(Refer to Part (b) of figure 1 for YXY's condition.)

As shown in figure 1, both Keil's $\sqrt{2}$-skeleton(S) and YXY's double-circle are symmetric with respect to edge $\overline{xy}$. YXY's condition also includes Gilbert's result (Gilbert, 1979) which stated that the shortest line segment among all the line segments with their endpoints in S belongs to any $MWT(S)$.

In many cases, an edge of $MWT(S)$ may not have symmetric geometric property required by sufficient conditions for identification. However, a simple extending of the known methods to asymmetric can run into difficulty. The difficulty caused by non-symmetric can be easily demonstrated by the following six-point set.

Figure 2 showed an example of six points such that $|\overline{xs}| + |\overline{xb}| \leq |\overline{rs}| + |\overline{rb}|$ and $|\overline{xy}| \leq |\overline{rb}|$, $|\overline{ra}|$, and $|\overline{rs}|$. Edge $\overline{xy}$ satisfies our sufficient conditions, however $\overline{xy}$ cannot be detected by any previous inclusion method. In part (a) of figure 2, vertex r lies inside the Keil's as well as YXY's empty circles, thus $\overline{xy}$ cannot be identified by these two methods.

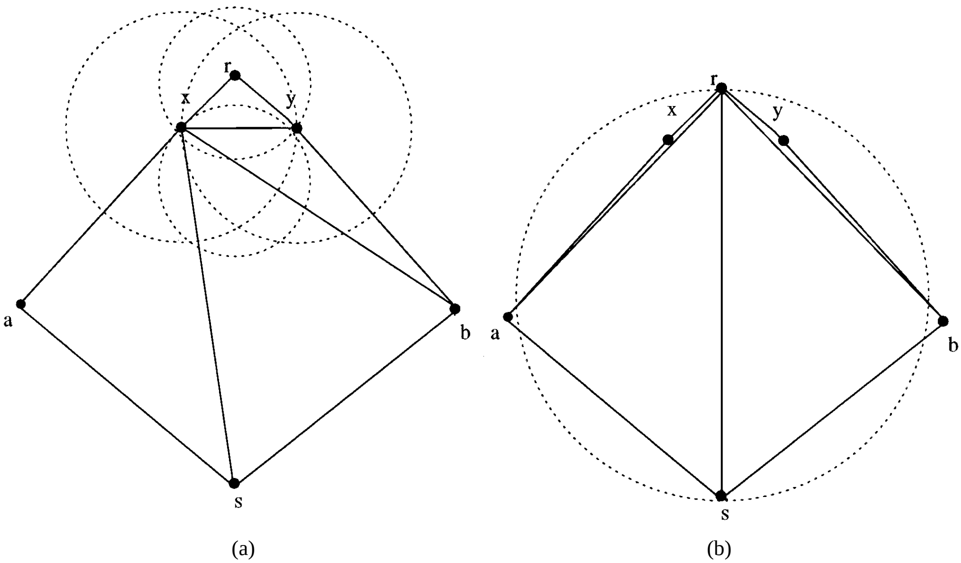


Figure 2. An example of a non-detectable edge for the known sufficient conditions.

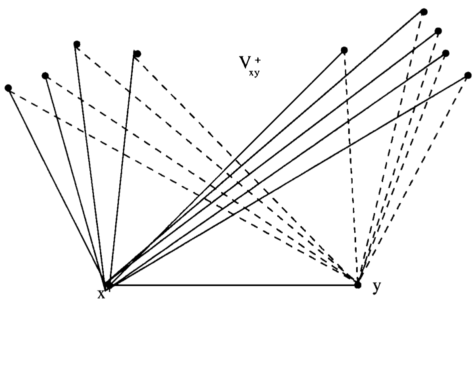
In part (b) of figure 2, $\overline{rs}$ is shorter than $\overline{ab}$, thus, they do not swap in quadraliteral $(arbs)$. Edge $\overline{ra}$ is shorter than $\overline{xs}$, thus, they do not swap in quadraliteral $(axrs)$. Edge $\overline{rb}$ is shorter than $\overline{ys}$, thus, they do not swap in quadraliteral $(sryb)$. Then, $\overline{rs}$ is an edge of a 4-*gon local optimal* triangulation. Hence, $\overline{xy}$ does not belong to the intersection of all 4-*gon local optimal* triangulations and cannot be identified by Dickerson and Montague's method.

To extend the local geometric property to asymmetric, it seems unavoidable to put further restriction on the 'neighboring' points of an identifying edge. We shall show these restrictions in the next section.

3. New sufficient conditions

We first give several definitions related to the local geometric of an edge $\overline{xy}$, then present the sufficient conditions.

Definition. Let $E(S)$ denote the set of all line segments with their endpoints in S. Let E_{xy} be the subset of $E(S)$, each of which crosses edge $\overline{xy}$ for $x, y \in S$. Let V_{xy} denote the endpoint set of E_{xy}, V_{xy}^{+} denote the subset of V_{xy} on the upper open halfplane bounded by the line extending $\overline{xy}$, and V_{xy}^{-} denote that on the lower closed halfplane, i.e., V_{xy}^{-} includes $\{x, y\}$. Then, V_{xy}^{+} is called **2star-shaped** if the interior of any triangle $\triangle xvy$ for every $v \in V_{xy}^{+}$ does not contain a vertex in V_{xy}^{+}. Similarly we can define 2*star-shaped* for V_{xy}^{-}. (Refer to part (a) of figure 3.) Let V_{xy}^{++} denote the subset of V_{xy}^{+} on the same side of x along the perpendicular bisector B_{xy} of $\overline{xy}$, and let V_{xy}^{+-} denote the subset on the same side as y. Let $EL_{v_i,v_j,y}$ denotes the ellipstic area specified by foci v_i and v_j with boundary point y. Let vertices $v_i \in V_{xy}^{++} \cup \{x\}$ and $v_j \in V_{xy}^{+-} \cup \{y\}$ be the foci (except simultaneously $v_i = x$ and $v_j = y$) and $|\overline{yv_i}| + |\overline{yv_j}|$ as the fixed sum of the lengths. V_{xy}^{+} is called **ellipse-disconnected** (w.r.t. V_{xy}^{-}) if for any v_i and v_j, $|\overline{xy}| < \min\{|\overline{xv_i}|, |\overline{yv_j}|\}$ and no vertex in V_{xy}^{-} is contained by the ellipses $EL_{v_i,v_j,y} \cup EL_{v_i,v_j,x}$ within the fan-area bounded by $\overrightarrow{v_iy}$ and $\overrightarrow{v_jx}$. (Refer to part (b) of figure 3, where only the empty ellipse area related to pair (v_i, v_j) is shown.) $V_{x,y}^{+}$



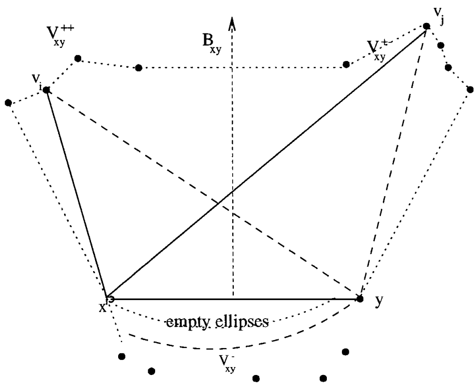


Figure 3. An illustration for the definitions.

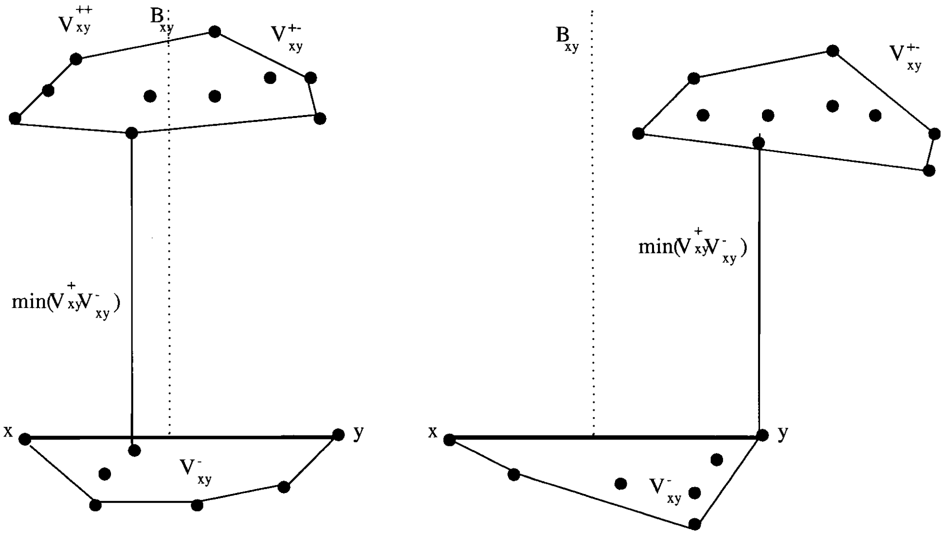


Figure 4. An illustration of a thin set.

is called **circle-disconnected** (w.r.t. V^-_{xy}) if for any $v \in V^+_{xy}$, $|\overline{xy}| < \min\{|\overline{xv}|, |\overline{yv}|\}$ and no vertex in V^-_{xy} is contained by the circle with v as center and with $\overline{vx}$ or $\overline{vy}$ as radius within the fan-area bounded by $\overrightarrow{vx}$ and $\overrightarrow{vy}$. A similar definition can be made for V^-_{xy}.

Definition. The *diameter* of V^-_{xy} (V^+_{xy}) is denoted by $d(V^-_{xy})$ ($d(V^+_{xy})$). Set V^+_{xy} is called **thin** if $d(V^+_{xy}) < d_{\min}(V^-_{xy}, V^+_{xy})$, where $d_{\min}(V^-_{xy}, V^+_{xy}) = \min\{\overline{v_p v_q} \mid v_p \in V^+_{xy}, v_q \in V^-_{xy}\}$. Similarly, define thin for V^-_{xy}. (Refer to figure 4 for the definition.)

Theorem 1. *Edge $\overline{xy}$ is in every $MWT(S)$ if* (1) *V^+_{xy} is* 2*star-shaped and ellipse-disconnected and* (2) $\omega(xy) = d(V^-_{xy})$. (*Refer to figure* 5.)

Proof: By contradiction. Suppose that $\overline{xy}$ does not belong to any $MWT(S)$. Then, there exists an $MWT(S)$ such that some of its edges, denoted by E_M ($\subseteq E_{xy}$), cross $\overline{xy}$. Let V^{+*}_{xy} denote the endpoint set of E_M belonging to V^+_{xy}, and V^{-*}_{xy}, belonging to V^-_{xy}. If we remove E_M from this $MWT(S)$, the resulting non-triangulated area, denoted R, is a connected region with $V^{+*}_{xy} \cup V^{-*}_{xy} \cup \{x, y\}$ as vertices. Since V^{+*}_{xy} is a subset of V^+_{xy}, V^{+*}_{xy} is also 2star-shaped and ellipse-disconnected, and since V^{-*}_{xy} is a subset of V^-_{xy}, $\omega(xy) = d(V^{-*}_{xy})$ is also hold. Note that the number of vertices contained in R is $|V^{+*}_{xy}| + |V^{-*}_{xy}| + 2$, which is $|E_M| + 3$. Then, the number of internal edges of R for any triangulation of R is also $|E_M|$. Now, we shall build a triangulation $T(R)$ such that $T(R)$ contains $\overline{xy}$ as an edge and $\omega(\mathrm{int}(T(R))$ is less than $\omega(E_M)$, where $\mathrm{int}(T(R)))$ is the internal edges of $T(R)$.

To do so, we add edges $\overline{v_i x}$ clockwisely at x and add edges $\overline{v_j y}$ anti-clockwisely at y, where $v_i \in V^{++*}_{xy}$ and $v_j \in V^{+-*}_{xy}$, and V^{++*}_{xy} and V^{+-*}_{xy} are the left subset and the right subset of V^{+*}_{xy} along B_{xy}, respectively; Add $\overline{xv'_j}$, where v'_j is the last vertex of V^{+-*}_{xy} in anti-clockwisely at y. Then, add $\overline{xy}$ so that the portion of R above the line extending $\overline{xy}$,

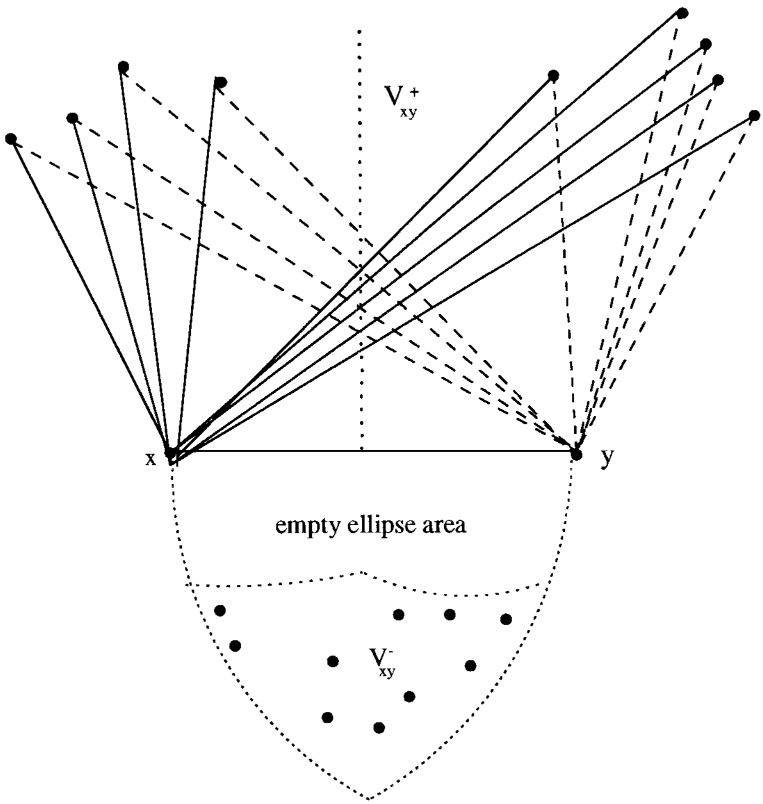


Figure 5. An illustration for sufficient conditions.

R^+, is triangulated. This can be realized because the 2star-shaped property of V_{xy}^{+*}. We further add edges to triangulate the portion of R below $\overline{xy}$, R^-, by any method. Thus, $T(R)$ is completed.

We only need compare the weight of the internal edges of $T(R)$ with the weight of E_M since the two triangulations of R shared the same boundary edges. The number of internal edges of $T(R)$ is equal to that of E_M, which allows us to build a match between them. If $|V_{xy}^{+*}| = 0$, we have done. If $|V_{xy}^{+*}| = 1$, then by condition (1), the edges of E_M ending at v_i (resp. v_j) is longer than $\overline{xv_i}$ (resp. $\overline{yv_j}$), and $\overline{xv_i}$ (resp. $\overline{yv_j}$) is longer than $\overline{xy}$, thus any edge of E_M is longer than $\overline{xy}$. Furthermore, by condition (2) any edge whose both endpoints are in V_{xy}^- is shorter than any edge in E_M. Thus, any edge in int$(T(R))$ is shorter than any edge in E_M, we have done. If $|V_{xy}^{+*}| \geq 2$, we shall consider two subcases: (a) one of V_{xy}^{++*} and V_{xy}^{+-*}, say V_{xy}^{++*}, is empty and (b) none of them is empty. In subcase (a), all these edges of int$(T(R))$ above $\overline{xy}$ are $\overline{v_jy}$ for $v_j \in V_{xy}^{+-*}$. By condition (1), every $\overline{v_jy}$ is shorter than these of E_M ending at v_j. Since there are $|V_{xy}^{+-*}| - 1$ such edges, the unmatched E_M is a size of $|V_{xy}^{-*}|$, i.e., there must have $|V_{xy}^{-*}|$ edges including $\overline{xy}$ to completely triangulate the remaining portion of R. Note by condition (2) that each of these added edges in R^- is shorter than or equal to any of E_M. Thus, $\omega(E_M) > \omega(\text{int}(T(R)))$. In subcase (b), there exists an edge of int$(T(R))$ crossing B_{xy}. Let it be $\overline{yv'_x}$. Let $\overline{v'_i v'_j}$ be the edge of the boundary of R^+ crossed by B_{xy}. We first match $\overline{yv'_i}$ and $\overline{yv'_j}$ with the two edges of E_M ending at v'_i and v'_j and sharing the other endpoint in V_{xy}^{-*}. This match is always possible because edge $\overline{v'_i v'_j}$ must belong to any $MWT(S)$, hence $\overline{v'_i v'_j}$ must belong to a triangle of $T(R)$ as well as a triangle of E_M of $MWT(S)$. By the empty ellipse property of condition (1), $\omega(|\overline{yv'_i}| + |\overline{yv'_j}|)$ is less than that of these two matched edges in E_M. For the remaining edges of int$(T(R))$, we match $\overline{v_i x}$ with an edge of E_M ending at v_i and $\overline{v_j y}$ with an edge of E_M ending at v_j (if it possible). It is implied by condition (1) that an edge of E_M ending at v_i (resp. v_j) is longer than $\overline{xv_i}$ (resp. $\overline{yv_j}$) (because the circle with v_i (v_j) as center and with $\overline{xv_i}$ ($\overline{yv_j}$) as radius in R^- is contained by the empty ellipse area). The remaining edges of int$(T(R))$ are matched with $\overline{xy}$ and those edges whose both endpoints in V_{xy}^{-*}. The remaining match is always possible as the same reason described in subcase (a). By condition (2) and the fact that any edge of E_M is longer than $\overline{xy}$, the weight of these remaining edges of int$(T(R))$ is less than or equal to that of the remaining edges in E_M. Thus, $\omega(\text{int}(T(R)))$ is less than $\omega(E_M)$, a contradiction. $\square$

(Refer to figure 6, which shows an example of the match between the two sets of internal edges of R in the proof. In this example, $V_{xy}^{+*} = \{1, 2, 3, 4\}$ and $V_{xy}^{-*} = \{5, 6, 7\}$. $E_M = \{a, b, c, d, e, f\}$. The dashed line segments are in the new triangulation $T(R)$, they are $\{\overline{x2}, \overline{x6}, \overline{y3}, \overline{y2}, \overline{y6}, \overline{xy}\}$. Note that $|c| + |d| > |\overline{y2}| + |\overline{y3}|$ due to the ellipse empty area property. One possible matching for the rest edges can be: $(\overline{x6}, a)$, $(\overline{y6}, f)$, $(\overline{x2}, b)$, and $(\overline{xy}, e)$).

Theorem 2. *Edge $\overline{xy}$ is in every $MWT(S)$ if* (1) *V_{xy}^+ is thin and circle-disconnected*, *and* (2) $\omega(xy) = d(V_{xy}^-)$.

Proof: Let the notations used in the following analysis be the same as those in the proof of Theorem 1. We shall build a new $T(S)$ with $\overline{xy}$ as an edge such that its weight is less than $MWT(S)$. If $|V_{xy}^{+*}| \leq 1$, then $\overline{xy}$ is in any $MWT(S)$ obviously. If $|V_{xy}^{+*}| \geq 2$, then we consider two subcases: (a) none of V_{xy}^{++*} and V_{xy}^{+-*} is empty and (b) one of them, say V_{xy}^{+-*}, is empty. In subcase (a), we shall traverse the boundary of R^+, clockwisely starting at the vertex next to x, to triangulate the area between the boundary of R^+ and the convex hull of V_{xy}^{++*}. Similarly, for V_{xy}^{+-*}. Add edges between the two convex chains to form $CH(V^{+*})$, and finally add edges from x or y to the vertices of $CH(V^{+*})$ accordingly to completely triangulate R^+. (Refer to Part (a) of figure 7 for the algorithm.) In subcase (b), we traverse the boundary of R^+, clockwisely starting at the vertex next to y, to triangulate the area between the boundary of R^+ and the convex hull of V_{xy}^{+-*}, then add edges from y to the convex hull of V_{xy}^{+-*} and add edges from x to the remaining vertices of $CH(V_{xy}^{+-*})$ to completely triangulate R^+. (Refer to Part (b) of figure 7 for the algorithm.) Finally, we use any method to triangulate R^-. Thus, the area determined by $V_{xy}^{+*} \cup V_{xy}^{-*} \cup \{x, y\}$ is completely triangulated. The above triangulation can always be done by our algorithm since R is a connected polygonal region that is weakly visible from $\overline{xy}$. Let us consider

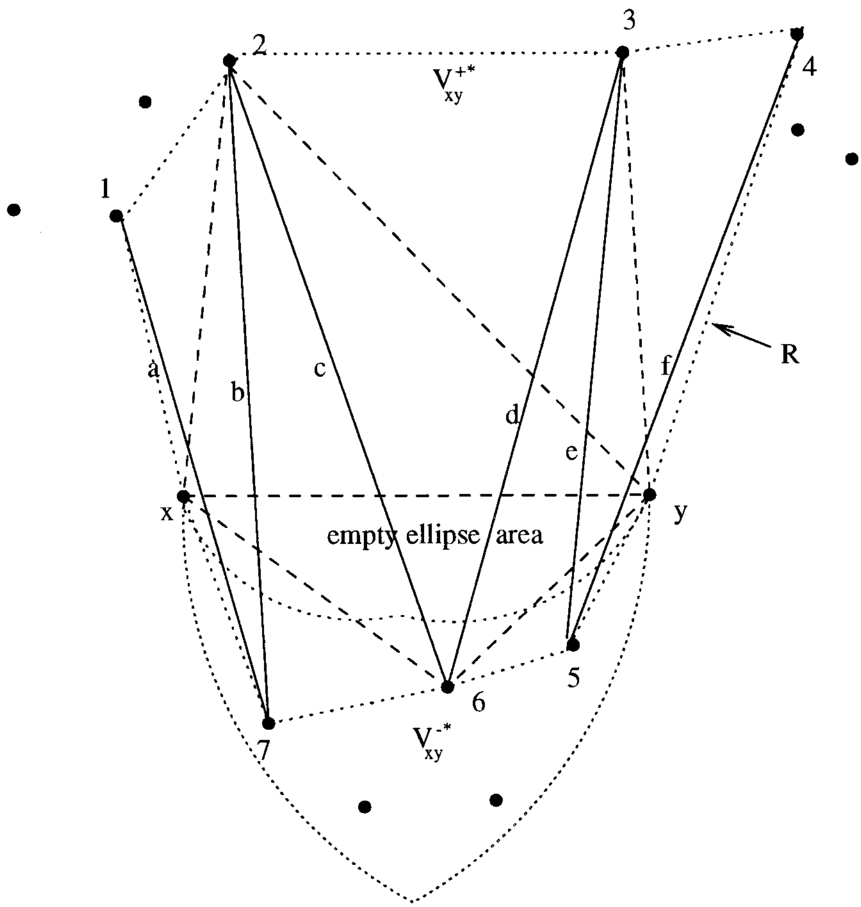


Figure 6. An illustration for the proof.

the weight of int$(T(R))$ and E_M. There are three types of the edges in $T(R)$: Type 1 edge has its both endpoints in V_{xy}^{+*} (resp. in V_{xy}^{-*}), Type 2 edge is either $\overline{xv_i}$ for $v_i \in V_{xy}^{++*}$ or $\overline{yv_j}$ for $v_j \in V_{xy}^{+-*}$, and Type 3 edge is either $\overline{yv_i}$ for $v_i \in V_{xy}^{++*}$ or $\overline{xv_j}$ for $v_j \in V_{xy}^{+-*}$. By the thin property of condition (1), a Type 1 edge is shorter than any edge of E_M and by the circle-disconnected property, an edge of Type 2 or Type 3 is also shorter than the edge of E_M ending at the same vertex of V_{xy}^{+}. Thus, the weight of int$(T(R))$ in R^+ is less than the weight of these matched edges of E_M. By condition (2), the weight of remaining edges of E_M is less than or equal to that of those edges of int$(T(R))$ in R^- including $\overline{xy}$. Thus, $\omega(\text{int}(T(R)))$ is less than $\omega(E_M)$, and hence the new triangulation $T(S)$ has a weight less than that of the original $MWT(S)$, a contradiction. Thus, $\overline{xy}$ must belong to any $MWT(S)$. $\square$

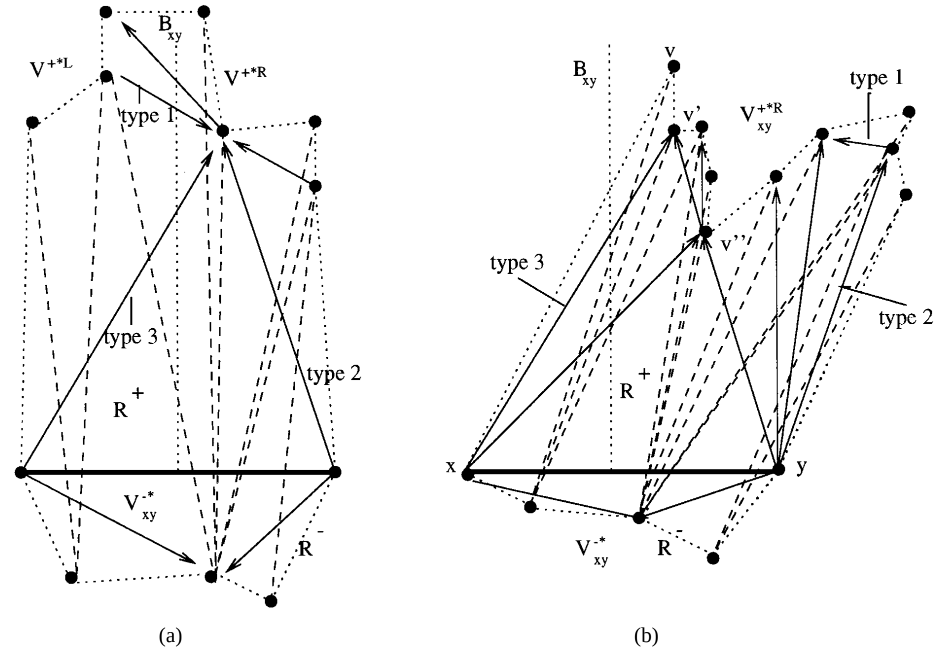


Figure 7. An illustration for the proof, where (v, v', v'', v''') is the convex hull of V^{+-*}.

4. The algorithm

SUB-MWT(S)

Input: S (a set of points in general position), $|S| = n$) and $E(S)$ (the set of all edges in S).

Output: **SUB-MWT(S)** (a subgraph of $MWT(S)$).

1. **SUB-MWT(S)** $\leftarrow \emptyset$.
2. Find $GT(S)$ and store the edges in E_{GT} by ascending length order.
3. While $E_{GT} \neq \emptyset$ Do
   (a) $e \leftarrow head(E_{GT})$; (* the shortest edge in current E_{GT} *)
   (b) Find the subset E_e of $E(S)$ crossed by e;
   (c) Call **Suff(**e, E_e**)**;
   (d) If **Suff(**e, E_e**)** $= 1$ Then **SUB-MWT(S)** $\leftarrow e$; Delete E_e.
4. EndDo.

Procedure Suff(*e*, E_e)

1. Let V^+ be these vertices of E_e lie on one halfplane bounded by the line extending e, and V^- be those vertices of E_e on the other halfplane. Sort V^+ by angular clockwisely at the left endpoint of e, and let V^{+l} denote this sequence; Sort V^+ by angular anti-clockwisely

at the right endpoint of e, and let V^{+r} denote this sequence. Sort V^- similarly and let the resulting sequences be V^{-l} and V^{-r}. Reverse V^{+r}, denote by $\overline{V^{+r}}$, and reverse V^{-r}, obtain $\overline{V^{-r}}$.
2. Find the diameters $d(V^+)$ and $d(V^-)$.
3. If V^{+l} is identical to $\overline{V^{+r}}$ and $d(V^-)$ is less than $\omega(e)$ and V^- lies outside the empty ellipse area A^- or V^{-l} is identical to $\overline{V^{-r}}$ and $d(V^+)$ is less than $\omega(e)$ and V^+ lies outside the empty ellipse area A^+, then **Suff(**e, E_e) $\leftarrow 1$;

End-Suff.

Lemma 1. *Algorithm* **SUB-MWT(S)** *produces a subgraph of* $MWT(S)$ *in* $O(n^3)$ *time and* $O(n^2)$ *space.*

Proof: The correctness of the algorithm is due to Theorem 1 and the fact implied by the sufficient condition that any edge produced by **SUB-MWT(S)** is shorter than any edge of E_M, thus it belongs to $GT(S)$. Let us consider the time complexity of the algorithm. Step 1 takes constant time. Step 2 takes $O(n^3)$ time by a trivial greedy method (for simplicity of the analysis). Step 3, the while-loop in **SUB-MWT(S)**, executes $O(n)$ times. Step (b) of the while-loop takes $O(n^2)$ time because there might have $O(n^2)$ line segments in E_e. The total time for this step in the entire algorithm is bounded by $O(n^3)$. Procedure **Suff** is called $O(n)$ times. Step 1 of **Suff** takes $O(n \log n)$ time due to the sortings. Step 2 takes $O(n \log n)$ time by first finding the convex hull and then finding the diameter. Step 3 takes $O(n^2)$ time to check the sufficient condition. That is, for each vertex v in V^-, check if v lies inside the empty ellipse area determined by vertices in V^+. We only need to test two ellipses: $EL_{v'_i,y,x}$ and $EL_{v'_j,x,y}$, where v'_i and v'_j are the two vertices closest to B_{xy} in the boundary R^+. This is because all other empty ellipse areas are contained by these two. It takes $O(n)$ time. Thus, the total time for procedure **Suff** in the entire algorithm is bounded by $O(n^3)$. Steps (a) and (b) do not exceed $O(n^2)$. The time complexity of the entire algorithm then follows. The Step (b) of Step 3 may yield $O(n^2)$ edges in E_e. The space complexity follows from $E(S)$ bounded by $O(n^2)$. □

Now, let us consider an algorithm for the second sufficient condition. Let the algorithm, denoted by **SUB-1-MWT(S)**, be the same as **SUB-MWT(S)** except replacing **Suff**(e, E_e) by **Suff** $-$ **1**(e, E_e).

Procedure Suff-1(e, E_e)

1. Find diameters $d(V^+)$ and $d(V^-)$, respectively.
2. Find $d_{\min}(V^+, V^-)$.
3. Test if V^+ is circle-disconnected w.r.t., V^- and vice versa.
4. If $d(V^+) < d_{\min}(V^+, V^-)$ and V^+ is circle-disconnected or $d(V^-) < d_{\min}(V^+, V^-)$ and V^- is circle-disconnected, then **Suff-1** $\leftarrow 1$; Else **Suff-1** $\leftarrow 0$

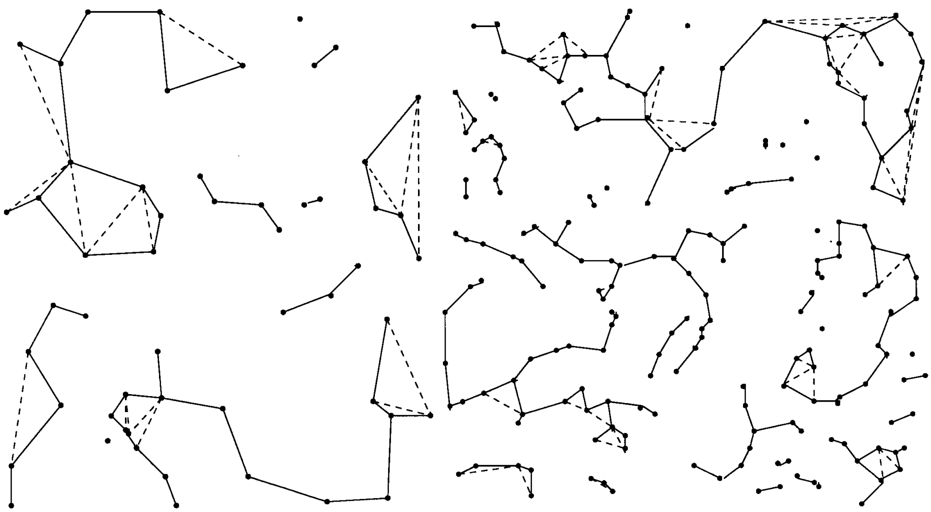
End-Suff-1.

Lemma 2. *Algorithm* **SUB-1-MWT(S)** *produces a subgraph of* $MWT(S)$ *in* $O(n^3)$ *time and* $O(n^2)$ *space.*

Proof: The correctness of the algorithm is due to Theorem 2. We only need consider the complexity of **Suff-1** since the rest is the same as in the proof of Lemma 1. Let $V = V^+ \cup V^-$. It takes $O(|V|)$ to identify V. It takes $O(|V| \log |V|)$ time to find the diameters of V^+ and V^- and the minimum distance between the two sets, $d_{\min}(V^+, V^-)$. Note that testing the circle-disconnected property of V^+ and V^- takes at most $O(|V^+| * |V^-|)$. Thus, the entire **Suff-1** takes $O(|V|^2)$ time and space. □

5. Concluding remarks

The new sufficient conditions for finding subgraphs of *MWT* in this paper are totally different from the previous known ones, which fall in two classes: (1) edges in all 4-*gon* local optimal triangulations and (2) β-skeleton and mutual nearest neighbors. Our conditions given in Section 3 are characterized with local non-symmetric geometric property. We have implemented an algorithm for identifying *MWT*-edges using the first sufficient condition. We try point sets of size 50 and 200. For each size, we take the average over 10 randomly generated point sets. We divided the algorithm in two steps. In the first step, we construct the β-skeleton of the set with $\beta = 1.17682$. In the second step, we delete all those edges crossing an edge of β-skeleton, and then test our condition. The results show that when the size is small (50), there is 22% increase w.r.t., the number of β-skeleton edges, and when the size is large (200), there is 15% increase. We observe that when size of point set becomes large, the 2*star-shaped* condition becomes difficult to be satisfied due to many long edges. We guess that if the exclusion condition (empty diamond area) is implemented in our algorithm, the situation may improve, (Refer to the following figure for some example.)



(a) (b)

Figure 8. Some example of experiment, where part (a) has 55 points and part (b) has 200 points. The solid lines are β-skeleton and dashed lines are new edges by our method.

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